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which reduces to $T = 2n\sqrt{Rr}$, the relation required; or putting $R = \frac{1}{2}d_1$, $r = \frac{1}{2}d_n$, we get $T = n\sqrt{d_1d_n}$.

If $n = 2$, we have the original theorem proposed by Mr. Collins.

The same method will apply, if the two original semicircles touched each other externally at A.

If we consider T the transverse, instead of the direct common tangent of O and N, then (2) becomes

$$d^2 = T_1^2 + (R + r)^2.$$

Substitute in (5) as before and

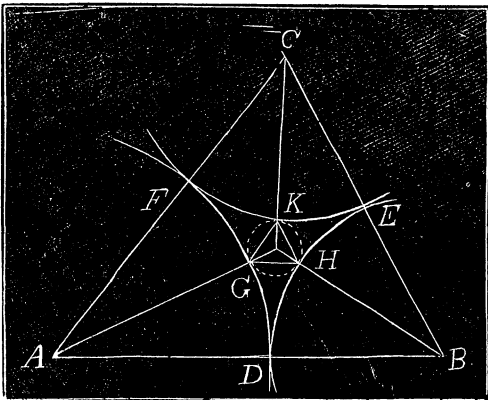
$$T_1 = 2\left((n^2 - 1)Rr\right)^{\frac{1}{2}} = \sqrt{(n^2 - 1)d_1d_n}.$$

In this case, if $n = 2$, $T_1^2 = 3d_1d_n$.

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*SOLUTION OF A PROBLEM.*

BY E. B. SEITZ, GREENVILLE, OHIO.

Three circles whose radii are  $a$ ,  $b$ ,  $c$ , touch each other externally. Within the space enclosed by them a circle is drawn tangent to the three circles, and within this circle three circles are drawn tangent to each other and to the three given circles. Calling the radii of these three circles  $x_1$ ,  $y_1$ ,  $z_1$ , we may determine three other circles, radii  $x_2$ ,  $y_2$ ,  $z_2$ , touching each other and the second set of circles in a similar way; and so on. Find the radii,  $x_n$ ,  $y_n$ ,  $z_n$ , of the  $n^{\text{th}}$  set of inscribed circles.



**SOLUTION.**—Let A, B, C be the centers of the given circles, GHK the circle which touches them, O its center, A', B', C' the centers of the first set of inscribed circles.

Put  $AD = a$ ,  $BD = b$ ,  $CF = c$ ,  $OG = r_1$ ,  $A'G = x_1$ ,  $B'H = y_1$ ,  $C'K = z_1$ ,  $r_0$  = the radius of the circle circumscribing the three given circles; also put  $\angle BAC = \beta$ ,

$\angle BAO = \theta$ ,  $\angle CAO = \phi$ ,  $\angle AOB = \omega$ .

$$\text{Then } \cos\beta = \frac{(a+b)^2 + (a+c)^2 - (b+c)^2}{2(a+b)(a+c)} = 1 - \frac{2bc}{(a+b)(a+c)},$$

$$\cos\theta = 1 - \frac{2br_1}{(a+b)(a+r_1)}, \cos\phi = 1 - \frac{2cr_1}{(a+c)(c+r_1)}, \text{ and}$$

$\cos(\theta + \phi) = \cos\beta$ , or by reduction

$$\cos^2\theta + \cos^2\phi - 2\cos\beta\cos\theta\cos\phi = 1 - \cos^2\beta \dots\dots\dots (1).$$

Substituting the values of  $\cos\beta, \cos\theta, \cos\phi$  in (1) and reducing, we have

$$[4abc(a+b+c) - (ab+ac+bc)^2]r_1^2 + 2abc(ab+ac+bc)r_1 = a^2b^2c^2; \text{ whence}$$

$$r_1 = \frac{\pm 2abc\sqrt{abc(a+b+c)} - abc(ab+ac+bc)}{4abc(a+b+c) - (ab+ac+bc)^2}$$

$$= \frac{abc}{\pm 2\sqrt{abc(a+b+c)} + ab+ac+bc}.$$

The first of these values is the value of  $r_1$ , and the other is that of  $r_0$ , taken with a contrary sign;

$$\therefore r_1 = \frac{abc}{2\sqrt{abc(a+b+c)} + ab+ac+bc},$$

$$\text{and } r_0 = \frac{abc}{2\sqrt{abc(a+b+c)} - (ab+ac+bc)}.$$

$$\cos\omega = 1 - \frac{2x_1y_1}{(r_1-x_1)(r_1-y_1)} = 1 - \frac{2ab}{(a+r_1)(b+r_1)}; \text{ whence}$$

$$\frac{x_1y_1}{(r_1-x_1)(r_1-y_1)} = \frac{ab}{(a+r_1)(b+r_1)} \dots\dots\dots (2),$$

$$\text{and similarly we find } \frac{x_1z_1}{(r_1-x_1)(r_1-z_1)} = \frac{ac}{(a+r_1)(c+r_1)} \dots\dots\dots (3),$$

$$\frac{y_1z_1}{(r_1-y_1)(r_1-z_1)} = \frac{bc}{(b+r_1)(c+r_1)} \dots\dots\dots (4).$$

From (2), (3), (4) we easily find

$$\frac{x_1}{r_1-x_1} = \frac{a}{a+r_1}, \quad \frac{y_1}{r_1-y_1} = \frac{b}{b+r_1}, \quad \frac{z_1}{r_1-z_1} = \frac{c}{c+r_1}; \text{ whence}$$

$x_1 = \frac{ar_1}{2a + r_1}, \quad y_1 = \frac{br_1}{2b + r_1}, \quad z_1 = \frac{cr_1}{2c + r_1}.$  Similarly we find

$x_2 = \frac{x_1 r_2}{2x_1 + r_2}, \quad y_2 = \frac{y_1 r_2}{2y_1 + r_2}, \quad z_2 = \frac{z_1 r_2}{2z_1 + r_2};$  and so on.

Taking the reciprocals of the values of  $x_1, x_2, x_3, x_4 \dots x_n$ , we have

$$\begin{aligned} \frac{1}{x_1} &= \frac{1}{a} + \frac{2}{r_1}, \\ \frac{1}{x_2} &= \frac{1}{x_1} + \frac{2}{r_2} = \frac{1}{a} + \frac{2}{r_1} + \frac{2}{r_2}, \\ \frac{1}{x_3} &= \frac{1}{x_2} + \frac{2}{r_3} = \frac{1}{a} + \frac{2}{r_1} + \frac{2}{r_2} + \frac{2}{r_3}, \\ &\dots\dots\dots \\ \frac{1}{x_n} &= \frac{1}{x_{n-1}} + \frac{2}{r_n} = \frac{1}{a} + \frac{2}{r_1} + \frac{2}{r_2} + \frac{2}{r_3} + \frac{2}{r_4} + \dots + \frac{2}{r_n} \dots\dots (5). \end{aligned}$$

Taking the reciprocals of the values of  $r_1$  and  $r_n$ , and subtracting,

we obtain  $\frac{1}{r_1} - \frac{1}{r_n} = \frac{2}{a} + \frac{2}{b} + \frac{2}{c} \dots\dots\dots (6).$

Similarly we find

$$\frac{1}{r_2} - \frac{1}{r_1} = \frac{2}{x_1} + \frac{2}{y_1} + \frac{2}{z_1} = \frac{2}{a} + \frac{2}{b} + \frac{2}{c} + \frac{12}{r_1} \dots\dots\dots (7).$$

Subtracting (6) from (7), and transposing, we get

$$\begin{aligned} \frac{1}{r_2} &= \frac{14}{r_1} - \frac{1}{r_n}; \text{ and similarly we find} \\ \frac{1}{r_3} &= \frac{14}{r_2} - \frac{1}{r_1}, \quad \frac{1}{r_4} = \frac{14}{r_3} - \frac{1}{r_2}, \dots\dots\dots \frac{1}{r_n} = \frac{14}{r_{n-1}} - \frac{1}{r_{n-2}}. \end{aligned}$$

Substituting the values of  $r_2, r_3, r_4 \dots r_{n-1}$ , taken from the preceding equations, and multiplying by  $(7 + 4\sqrt{3}) + (7 - 4\sqrt{3})$  instead of 14, we have

$$\begin{aligned} \frac{1}{r_1} &= \frac{\frac{1}{8\sqrt{3}} [(7+4\sqrt{3})-(7-4\sqrt{3})]}{r_1}, \\ \frac{1}{r_2} &= \frac{\frac{1}{8\sqrt{3}} [(7+4\sqrt{3})^2-(7-4\sqrt{3})^2]}{r_1} - \frac{\frac{1}{8\sqrt{3}} [(7+4\sqrt{3})-(7-4\sqrt{3})]}{r_n}, \end{aligned}$$

$$\frac{1}{r_3} = \frac{\frac{1}{8\sqrt{3}} [(7+4\sqrt{3})^3 - (7-4\sqrt{3})^3]}{r_1} - \frac{\frac{1}{8\sqrt{3}} [(7+4\sqrt{3})^2 - (7-4\sqrt{3})^2]}{r_0},$$

.....

$$\frac{1}{r_n} = \frac{\frac{1}{8\sqrt{3}} [(7+4\sqrt{3})^n - (7-4\sqrt{3})^n]}{r_1} - \frac{\frac{1}{8\sqrt{3}} [(7+4\sqrt{3})^{n-1} - (7-4\sqrt{3})^{n-1}]}{r_0}.$$

Adding these equations, we get

$$\begin{aligned} & \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \dots \dots \dots \frac{1}{r_n} \\ &= \frac{\frac{1}{48} [(7+4\sqrt{3})^{n+\frac{1}{2}} + (7-4\sqrt{3})^{n+\frac{1}{2}}]}{r_1} - \frac{\frac{1}{48} [(7+4\sqrt{3})^{n-\frac{1}{2}} + (7-4\sqrt{3})^{n-\frac{1}{2}}]}{r_0} \\ &= \left\{ \frac{1}{4\sqrt{3}} [(7+4\sqrt{3})^n - (7-4\sqrt{3})^n] \sqrt{abc(a+b+c)} + \frac{1}{12} [(7+4\sqrt{3})^n + (7-4\sqrt{3})^n] (ab+ac+bc) \right\} \div abc. \end{aligned}$$

Substituting this value in (5), and taking reciprocals, we find

$$x_n = abc \div \left\{ \frac{1}{2\sqrt{3}} [(7+4\sqrt{3})^n - (7-4\sqrt{3})^n] \sqrt{abc(a+b+c)} + \frac{1}{6} [(7+4\sqrt{3})^n + (7-4\sqrt{3})^n - 2] (ab+ac) + \frac{1}{6} [(7+4\sqrt{3})^n + (7-4\sqrt{3})^n + 4] bc \right\}.$$

Similarly we find

$$y_n = abc \div \left\{ \frac{1}{2\sqrt{3}} [(7+4\sqrt{3})^n - (7-4\sqrt{3})^n] \sqrt{abc(a+b+c)} + \frac{1}{6} [(7+4\sqrt{3})^n + (7-4\sqrt{3})^n - 2] (ab+bc) + \frac{1}{6} [(7+4\sqrt{3})^n + (7-4\sqrt{3})^n + 4] ac \right\}.$$

$$z_n = abc \div \left\{ \frac{1}{2\sqrt{3}} [(7 + 4\sqrt{3})^n - (7 - 4\sqrt{3})^n] \sqrt{abc(a+b+c)} + \frac{1}{6} [(7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n - 2](ac + bc) + \frac{1}{6} [(7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n + 4]ab \right\}.$$

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NOTE ON THE BINOMIAL THEOREM.

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BY GAETANO LANZA, ASSISTANT PROF. OF MATH. AND MECH., MASS. INSTITUTE OF TECHNOLOGY.

MR. EDITOR:—Allow me to call your attention to a demonstration of the Binomial Formula for positive integral exponents, given by Bobillier, which, though very simple and lucid, is not very generally known.

It does not involve at all the subjects of Permutations and Combinations, and hence in order to understand the Binomial Theorem the student is not obliged previously to master a number of propositions on the above subjects.

It would therefore seem very desirable that this demonstration should find a place in our text books and be taught in our schools.

For the benefit of any of your readers who may not be familiar with the notation employed, I will state that $\underline{n}$ (*=factorial n*) is an abbreviated way of writing $1.2.3.4. n$, that is the product of the natural numbers from 1 to n inclusive; thus $\underline{3} = 1.2.3 = 6$; $\underline{4} = 1.2.3.4 = 24$, &c.

The demonstration is as follows:

The following three equations will be evident on inspection, viz.:

$$\begin{aligned} \frac{(x+a)^1}{\underline{1}} &= \frac{x}{\underline{1}} + \frac{a}{\underline{1}}, \\ \frac{(x+a)^2}{\underline{2}} &= \frac{x^2}{\underline{2}} + \frac{x}{\underline{1}} \cdot \frac{a}{\underline{1}} + \frac{a^2}{\underline{2}}, \\ \frac{(x+a)^3}{\underline{3}} &= \frac{x^3}{\underline{3}} + \frac{x^2}{\underline{2}} \cdot \frac{a}{\underline{1}} + \frac{x}{\underline{1}} \cdot \frac{a^2}{\underline{2}} + \frac{a^3}{\underline{3}}. \end{aligned}$$

It is now proposed first to find an expression equal to

$$\frac{(x+a)^n}{\underline{n}}$$